Comprehensive Examination

Math 650-651 / Optimization / January 2013

Name ______________________________

INSTRUCTIONS: (i) You must **complete** Problem 1, 4, and 5. You must choose: **one** problem from the set $\{2, 3\}$, and **one** problem from the set $\{6, 7\}$. Please *mark* clearly which problem you would like to be graded from the set $\{2, 3, 6, 7\}$.

1. **[20 pts]** Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$, we have
$$f((1-\lambda)x + \lambda y) \;\le\; (1-\lambda)f(x) + \lambda f(y).$$
   (a) Prove that a differentiable (say Frechet) f is convex if and only if for any $x, y \in \mathbb{R}^n$,
$$f(y) \;\ge\; f(x) + \langle \nabla f(x), y - x\rangle.$$
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, and let $C \subseteq \mathbb{R}^n$ be a closed convex set. Show that a point $x^* \in C$ minimizes f on C if and only if the following *variational inequality* holds true:
$$\langle \nabla f(x^*), x - x^*\rangle \;\ge\; 0 \quad \text{for all } x \in C.$$

2. **[15 pts]** *Motzkin's Transposition Theorem* characterizes when a general linear inequality system
$$Ax < a, \quad Bx \le b, \quad Cx = c \qquad (I)$$
is inconsistent, that is, its solution set is empty. Here A, B, C have the same number of columns.
   (a) State this theorem (but not prove!): What is the "dual statement" statement (II) in Motzkin's Theorem corresponding to (I), and what is the exact statement connecting (I) and (II)?
   (b) Now, consider the statement
$$x > 0, \; Bx \le b. \qquad (III)$$
   Use Motzkin's Theorem in (a) to show that if (III) is inconsistent, then either there exists y such that
$$B^T y \ge 0, \; \langle b, y\rangle < 0, \; y \ge 0, \qquad (IV)$$
   or there exists y such that
$$B^T y \ge 0, \; B^T y \ne 0, \langle b, y\rangle = 0, \; y \ge 0. \qquad (V)$$

3. **[15 pts]** Consider the optimization problem
$$\max\left\{\prod_{i=1}^{n} x_i : \sum_{i=1}^{n} x_i = 1, x_i \ge 0, i = 1, \ldots, n\right\}.$$
   (a) Determine the global maximizer(s) of the problem by solving the above nonlinear program using constrained optimization techniques (FJ, KKT conditions, etc.). Show all your work.

(b) Use your results in (a) to show the arithmetic-geometric mean inquality

$$\sqrt[n]{x_1 \cdots x_n} \;\le\; \frac{x_1 + \cdots + x_n}{n},$$

with equality holding if and only of all x_i are equal, that is, $x_1 = x_2 \ldots = x_n$.

4. **[25 pts]** Consider the optimization problem

$$\begin{array}{ll} \max & x_1^2 + x_2 \\ \text{s. t.} & x_1^2 + x_2^2 - 9 \le 0 \\ & x_1 + x_2 - 1 \le 0 \end{array}$$

(a) Sketch the feasible region and the level curves of the objective function.

(b) Explain, giving a theoretical argument, that any local maximizer must satisfy the KKT conditions.

(c) State the KKT (Karush–Kuhn–Tucker) conditions for the optimization problem above. Then, determine which of the following four points satisfy the KKT conditions:

$$(i) \quad (x_1^*, x_2^*) = \left(\frac{-\sqrt{35}}{2}, \frac{1}{2}\right), \qquad (ii) \quad (x_1^*, x_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right),$$

$$(iii) \quad (x_1^*, x_2^*) = \left(1 + \frac{\sqrt{17}}{2}, 1 - \frac{\sqrt{17}}{2}\right), \qquad (iv) \quad (x_1^*, x_2^*) = \left(1 - \frac{\sqrt{17}}{2}, 1 + \frac{\sqrt{17}}{2}\right).$$

(d) Use the second order test to determine which of the four points in (b) are local maximum points. Make sure to give a concise statement of the test(s) you are using.

5. **[20 pts]** In the quadratic program $\min\{\frac{1}{2}\langle Qx, x\rangle + \langle c, x\rangle \,:\, Ax \le b\}$, Q is an $n \times n$ symmetric, positive definite matrix.

(a) Write the Lagrangian function and use it to determine the dual program.

(b) In particular, show that the dual program can be written in the form $\max\{\frac{1}{2}\langle Ry, y\rangle + \langle d, y\rangle : y \ge 0\}$, where R is an $n \times n$ symmetric matrix and $d \in \mathbb{R}^n$.

(c) State the strongest *duality theorem(s)* that apply to this primal-dual pair of quadratic programs.

(d) *(Extra Credit 4 pts.)* Suppose that you know an optimal solution y^* to the dual program. Does this help to determine an optimal solution x^* to the primal program. Is there an explicit formula that expresses x^* as a function of y^*? If, so, give it.

6. **[20 pts]** Consider the problem $\min 5x^2 + 5y^2 - xy - 11x + 11y + 11$.

(a) Find a point satisfying the first-order necessary conditions for a solution.

(b) Show that this point is a global minimum.

(c) What would be the rate of convergence of steepest descent for this problem?

(d) Starting at $(x_0, y_0) = (0, 0)$, how many steepest descent iterations would it take (at most) to reduce the function value to 10^{-11}?

(e) Starting at $(x_0, y_0) = (0, 0)$, compute the next iterate (x_1, y_1).

7. **[20 pts]** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map. Suppose we want to solve the equation system $F(x) = 0$ using Newton's method.

(a) Give a derivation of Newton's iteration formula which, given the current approximate root vector $x_k \in \mathbb{R}^n$, computes the next iterate x_{k+1}.

(b) Give a discussion of the convergence of Newton's method, and its convergence rate (including the conditions needed for the converge rate to apply).

(c) Now the nonlinear equation system

$$\begin{aligned} &2(x+y)^2 + (x-y)^2 - 8 = 0 \\ &5x^2 + (y-3)^2 - 9 = 0. \end{aligned}$$

(d) Note that $(x^*, y^*) = (1, 1)$ is a root of the system. Starting at $(x_0, y_0) = (2, 0)$, compute the next iterate (x_1, y_1).

(e) Assuming that Newton's method converges in this case, what would be the asymptotic convergence rate of Newton's method in this case? Justify your answer.