COMPREHENSIVE EXAMINATION

Math 650 / Optimization / August 2009

Name ____

INSTRUCTIONS: (i) Choose **one** problem from the set $\{1, 2\}$ (40 points), and **two** problems from the set $\{3, 4, 5\}$ (60 points). Please *mark* clearly which problems you would like to be graded!

1. Consider the optimization problem

$$\begin{array}{rl} \min & xyz \\ \text{subject to} & x + y + z = 0, \\ & x^2 + y^2 + z^2 = 1. \end{array}$$

Find the solution of this nonlinear program by completing the following, incremental, steps.
(a) Give a theoretical reason as to why every local minimizer must satisfy the KKT conditions.
(b) Form the Lagrangian function

$$L(x, y, z, \lambda, \mu) = xyz + \lambda(x + y + z) + \mu(x^2 + y^2 + z^2 - 1),$$

where λ and μ are the multipliers; write the KKT conditions.
(c) Use the KKT conditions to show that $3xyz = -2\mu$, and argue that we must have $\mu > 0$.
Hint: We are minimizing xyz!
(d) Use the KKT conditions to show that

$$x(\lambda + 2\mu x) = y(\lambda + 2\mu y) = z(\lambda + 2\mu z) = -xyz = \frac{2\mu}{3},$$

and argue that x, y, z must be the roots of the equation

$$u^2 + \gamma u - \frac{1}{3} = 0, \tag{1}$$

where $\gamma = \frac{\lambda}{2\mu}$.
(e) Using **(c)** and the KKT conditions, argue that if (x^*, y^*, z^*) is an optimal solution with $x^* \leq y^* \leq z^*$, then $x^* < 0 < y^* \leq z^*$. Then use **(d)** to show that $y^* = z^*$.
(f) Using (1) argue that $y^* = z^* = -(x^* + y^*) = \gamma$, and that $x^* y^* = -1/3$ so that $x^* = \frac{-1}{3\gamma}$.
(g) Use $x^* + y^* + z^* = 0$ to show that $y^* = z^* = \gamma = \frac{1}{\sqrt{6}}$, and $x^* = \frac{-2}{\sqrt{6}}$.

2. Consider the optimization problem

$$\begin{array}{rl} \max & x^2 + y^2 \\ s.t. & x^2 - y^2 \geq 1 \\ & x \leq 3 \end{array}$$

(a) Sketch the feasible region.
(b) Write the Fritz John conditions for the critical point(s) of the problem.
(c) Show that every Fritz John point must satisfy the KKT conditions.
(d) Determine which of the following three points satisfy the KKT conditions: **A.** $(-1, 0)$, **B.** $(3, 0)$, **C.** $(3, -2\sqrt{2})$.
(e) Determine whether **C.** is a local maximizer; use second order conditions for this purpose.

3. Answer fully *two* of the following three, **unrelated** questions, clearly indicating your choices.

(a) State Jensen's inequality for a convex function $f : \mathbb{R}^n \to \mathbb{R}$. Assuming its truth, state and prove the characterization of equality in Jensen's inequality when f is a strictly convex function.

(b) Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and denote by $\Pi_C(x)$ the projection of x onto C. (That is, $\Pi_C(x)$ is the unique solution to the problem $\min\{||z - x||^2 : z \in C\}$.) State the *variational inequality* which characterizes Π_C, and use it to prove that π_C is non–expansive, that is,

$$||\pi_C(x) - \pi_C(y)|| \le ||x - y||, \qquad x, y \in \mathbb{R}^n.$$

(c) Let $P_1, P_2 \subseteq \mathbb{R}^n$ be two convex polyhedra. Prove that the Minkowski sum $P_1 + P_2$ is also a convex polyhedron.

4. Consider the set $K := \{x : Ax < 0\}$ where A is an $m \times n$ matrix.

(a) Show, by elementary arguments (that is, using no convexity), that $K = \emptyset$ if and only if

$$\{y \in \mathbb{R}^m : y_i < 0, i = 1, \ldots, m\} \cap \{Ax : x \in \mathbb{R}^n\} = \emptyset. \qquad (1)$$

(b) Assuming (1) is true, show by a separation argument, that there exists $a \in \mathbb{R}^m$ satisfying

$$a \ge 0, \quad a \ne 0, \quad A^T a = 0. \qquad (2)$$

(c) Combine **(a)** and **(b)** to prove that $\{x : Ax < 0\} = \emptyset$ if and only if the zero vector is in the convex hull of the rows of A.

5. Answer the following, **unrelated**, questions.

(a) Consider the "diamond" D in the plane with vertices at the points $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Describe D by four linear inequalities; use this to determine the *polar* D of, D^* where

$$D^* = \{y \in \mathbb{R}^2 : \langle x, y\rangle \le 1, \ \forall x \in D\}.$$

Hint: use Farkas Lemma.

(b) Formulate the following min-max problem as a linear program:

$$\min_{x \in \mathbb{R}^n} \max_{1 \le i \le m} \ a_i^T x + b_i.$$