COMPREHENSIVE EXAMINATION

Math 650 / Optimization / January 2007
(Prepared by Dr. O. Güler)

Name ____

INSTRUCTIONS: (i) You *must* solve
(i) *One* problem from the problem set **1, 2** *(33 points)*;
(ii) *Problem 3 (34 points)*; and
(iii) *One* problem from the set **4, 5** *(33 points)*.
Please *mark* clearly which problems you would like to be graded. (Otherwise, Problems 2, 3, and 4 will be graded.)

1. Consider the optimization problem

$$\begin{aligned} \min \quad & \sum_{i=1}^n f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = 1, \quad (P) \\ & x_i \geq 0, i = 1, \ldots, n. \end{aligned}$$

Suppose that a point $x^* = ((x_1^*, \ldots, x_n^*) \geq 0$ solves the above problem. Define $g_i := f_i'(x_i^*)$. Show that there exists a scalar k such that

$$g_i \geq k, \text{ and } (g_i - k)x_i^* = 0, \qquad \text{for } i = 1, \ldots, n.$$

2. Define

$$e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n, \quad e_1 = (1, 1, \ldots, 1) \in \mathbb{R}^n, \quad c = e/n = (1/n, 1/n \ldots, 1/n)^T \in \mathbb{R}^n.$$

Consider the optimization problem

$$\begin{aligned} \min \quad & y_1 = \langle e_1, y \rangle \\ \text{s.t.} \quad & ||y - c||^2 \leq \frac{1}{n(n-1)}, \quad (Q) \\ & \langle e, y \rangle = 1. \end{aligned}$$

(a) Write the KKT optimality conditions.

(b) Verify that $\left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)^T$ is an optimal solution to (Q).

(c) Prove that the solution in (b) is the unique optimal solution to (Q).

3. Consider the optimization problem

$$\min\{-\sum_{i=1}^n \ln x_i : \sum_{i=1}^n x_i = 1, \quad (P).$$

(a) Show that the optimal solution is $x^* = (1/n, \ldots, 1/n)^T$.

(b) Formulate the dual (D) of problem (P). Find the dual optimal solution λ^*. State the strong duality theorem and justify rigorously whether it is true in this case.

(c) Recall that the arithmetic–geometric mean inequality (AGM) is

$$\prod_{i=1}^n x_i^{1/n} \leq \frac{\sum_{i=1}^n x_i}{n}, \quad x_i > 0, i = 1, \ldots, n.$$

Show how to prove (AGM) using part (a).

4. Consider the cone $K \subset \mathbb{R}^3$ whose points satisfy the linear inequalities

$$\begin{aligned} x_1 + x_2 - x_3 &\leq 0, \\ x_3 &\geq 0, \\ 2x_1 - x_2 + x_3 &\geq 0. \end{aligned}$$

(a) Give the definition of the dual (polar) cone of a general convex cone C.

(b) Use Farkas Lemma or otherwise to explicitly compute the dual cone of K above.

5. (a) Give the definition of a *strictly* convex function.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function, and $x_i \in \mathbb{R}^n$, $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. If

$$f(\lambda_1 x_1 + \ldots + \lambda_k x_k) = \lambda_1 f(x_1) + \ldots + \lambda_k f(x_k),$$

then show that $x_1 = x_2 = \ldots = x_k$.