# COMPREHENSIVE EXAMINATION

Math 650 / Optimization / August 2006
(Prepared by Dr. O. Güler)

Name ____________________

INSTRUCTIONS: (i) You *must* solve *either* **Problem 1** *or* **both Problems 2 and 3** *(60 points)*; (ii) *one* problem from the set $\{4, 5\}$ (40 points). Please *mark* clearly which problems you would like to be graded - otherwise, Problems 1 and 5 will be graded.

**1.** Consider the optimization problem (P)

$$\begin{array}{ll} \min & xyz \\ \text{s.t.} & x^2+y^2+z^2 \le 1, \\ & x+y+z=1. \end{array}$$

(a) Write the Lagrangian function

$$L(x,y,z;\lambda_0,\lambda,\mu) := \lambda_0 xyz + \frac{\lambda}{2}(x^2+y^2+z^2-1) + \mu(x+y+z-1).$$

Write down the FJ (Fritz John) conditions for (P), which are necessary for a local minimizer $(x^*, y^*, z^*)$ of (P). Show that $\lambda_0 \neq 0$, either by citing an appropriate constraint qualification rule (preferred), or by an explicit, ad–hoc reasoning.

(b) Write down the KKT conditions for (P) which must be satisfied at all local minimizers of (P).

(c) Consider the following three points: $\{A(1/3, 1/3, 1/3),\ B(0, 0, 1),\ C(2/3, 2/3, -1/3)$. Determine, with full justification, which of these points satisfy the KKT conditions.

(d) Use second order necessary/sufficient conditions to determine whether the point $C$ is a local minimizer.

(e) *(Extra Credit, 6 pts.)* Use the equations $yz + \lambda x + \mu = 0$ and $xz + \lambda y + \mu = 0$ appearing in the KKT conditions to conclude that

$$\text{either } x = y, \text{ or } z = \lambda.$$

Show that similar conditions equalities must also be true for the variable pairs $\{x, z\}$ and $\{y, z\}$. Prove that these imply that it is impossible to have all three variables $x, y, z$ mutually distinct, that is, at least two of the three variables $x, y, z$ must be the same, say $x = y$.

**2.** Consider the optimization problem (P): $\min\{f(x) : x \in C\}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, say with continuous gradient $\nabla f(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)^T$, and $C \subseteq \mathbb{R}^n$ is a closed, convex set.

(a) Show that if $x^* \in C$ is a local minimizer of $P$, then

$$\langle \nabla f(x^*), x - x^* \rangle \ge 0, \quad \forall x \in C. \quad (1)$$

(Recall that $\langle x, y\rangle = x^T y$ is the inner product in $\mathbb{R}^n$.)

(b) Show that if $C = \{x : Ax = b\}$ is an affine set, where $A$ is an $m \times n$ matrix, then (1) is equivalent to the condition that $\nabla f(x^*)$ is orthogonal to $N(A)$, the null space of $A$. Finally, show that there exists $y$ such that $\nabla f(x^*) = A^T y$.

(c) If $f$ is a convex function and $x^*$ satisfies (1), then show that $x^*$ is a global minimizer of $f$ on $C$.

**3.** Consider the quadratic function $f(x) := \frac{1}{2}x^T Ax + c^T x + \alpha$ on $\mathbb{R}^n$, where $A$ is *any* symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Suppose that $f(x) \geq 0$, that is, $f$ is non–negative on $\mathbb{R}^n$. Assuming $A$ is a diagonal matrix, prove the following:

(a) Show that $A$ is positive semi–definite, that is, all diagonal elements of $A$ are non–negative.

(b) Show that $f$ is a convex function.

(c) Show that $f$ achieves its infimum, that is, $f$ has a global minimizer.

(d) Now assume that $A$ is an arbitrary symmetric $n \times n$ matrix. Extend the proof of (a)–(c) for this general case. *Hint:* reduce to the diagonal case by diagonalizing $A$!

**4.** *Motzkin's Transposition Theorem* states that the following: Let $A, B, C$ be matrices with the same number of rows. Then exactly one of the following systems is consistent:

$$A^T x < 0, \quad B^T x \leq 0, \quad C^T x = 0, \qquad (I)$$

$$Ay + Bz + Cw = 0, \quad y \geq 0, \quad y \neq 0, \quad z \geq 0. \qquad (II)$$

The following alternative theorem of *von Neumann–Morgenstern* plays an important role in game theory: Let $D$ be an $n \times m$ matrix. Either there exists a vector $x \in \mathbb{R}^m$ satisfying

$$\sum_{j=1}^{m} d_{ij} x_j \leq 0, \ x \geq 0, \ \sum_{j=1}^{m} x_j = 1, \qquad (1)$$

or there exists a vector $y \in \mathbb{R}^n$ satisfying

$$\sum_{i=1}^{n} d_{ij} y_i > 0, \ y \geq 0, \ \sum_{i=1}^{n} y_i = 1, \qquad (2)$$

but not both.

Show, by straightforward manipulation, that von Neumann and Morgenstern follows directly from Motzkin's Transposition Theorem.

**5.** Consider the problem of projecting a point $a \in \mathbb{R}^n$ onto the unit simplex. Write the problem as the optimization problem

$$\min\{\frac{1}{2}||x - a||^2 : \langle e, x\rangle = 1, x \geq 0\}, \quad (P).$$

Write the Lagrangian

$$L(x, \lambda, \mu) := \frac{||x - a||^2}{2} - \langle \lambda, x\rangle + \mu(\langle e, x\rangle - 1).$$

(a) Show that the primal problem is indeed precisely the minimax problem

$$\min_{x \in \mathbb{R}^n} \max_{0 \leq \lambda \in \mathbb{R}^n, \mu \in \mathbb{R}} L(x, \lambda, \mu).$$

(b) Show that the (Lagrangian) dual of (P) is the problem

$$\max_{0 \leq \lambda \in \mathbb{R}^n, \mu \in \mathbb{R}} -\frac{1}{2}||\mu e - \lambda - a||^2 - \mu + \frac{1}{2}||a||^2. \quad (D)$$

*Hint:* verify that

$$L(x, \lambda, \mu) = \frac{1}{2}||x - a + \mu e - \lambda||^2 - \frac{1}{2}||\mu e - \lambda - a||^2 - \mu + \frac{1}{2}||a||^2.$$

(c) What does the *Strong Duality Theorem (SDT)* say about the problem pair (P)–(D)? State it, and show that SDT holds true, citing an appropriate theorem if necessary.