COMPREHENSIVE EXAMINATION
Math 650 / Optimization / January 2006
(Prepared by Dr. O. Güler)

Name ______________________________

INSTRUCTIONS:
You *must* solve *one* problem from the set $\{1, 2\}$ (50 points), and *one* problem from the set $\{3, 4\}$ (50 points), but please *mark* clearly which *two* problems you would like to be graded. Otherwise, problems 1 and 4 will be graded.

1.(a) Consider the quadratic function $f(x) := \frac{1}{2}x^T Ax + c^T x + \alpha$ on $\mathbb{R}^n$, where A is **any** symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

(a) Give the definitions for a local minimizer of f and a global minimizer of f.

(b) Write the first and second order necessary conditions for a local minimizer $x^* \in \mathbb{R}^n$ of f.

(c) Show that if x^* is a local minimizer of f, then x^* is actually global minimizer of f on $\mathbb{R}^n$.

(d) Now consider the optimization problem

$$\min\{f(x) : a^T x = \beta\} \quad (P),$$

that is, the minimization of f on an hyperplane $\{x : a^T x = \beta\}$, where $a \in \mathbb{R}^n$ is a non–zero vector.
Write the first and second order necessary conditions for a local minimizer x^* of (P).

(e) Show that if x^* satisfies the first and second order necessary in part (d), then x^* is actually global minimizer of (P).

2. Consider the problem (P)

$$\begin{array}{ll} \min & xy \\ \text{s.t.} & x^2 + y^2 \le 2, \\ & x + y \ge 0. \end{array}$$

(a) Write down the FJ (Fritz John) conditions for (P), which are necessary for a local minimizer (x^*, y^*) of (P). Show that $\lambda_0 \neq 0$, that is, KKT (Karush–Kuhn–Tucker) conditions are satisfied at all points satisfying the FJ conditions.

(b) Write the resulting KKT conditions for a local minimizer of (P).

(c) Consider the following four points: $\{(1,-1),(-1,1),(1,1),(0,0)\}$. Determine, with full justification, which of these points satisfy the KKT conditions.

(d) Use second order necessary/sufficient conditions to determine which KKT point(s) is a local minimizer. Is there any local minimizer which is not global minimizer?

3. Consider the convex programming problem (P), $\min\{-\ln x - \ln y : x^2 + y \leq 1\}$.

(a) Formulate the Lagrange dual (D) of (P) in a single variable λ in such a way that no primal variables x, y appear in (D).

(b) State the Strong Duality Theorem as it applies to the pair (P) and (D). Prove that either this duality theorem holds true, or argue that it does not hold true.

(c) Solve the dual problem (D), that is, compute its optimal solution λ^*.

(d) Use the numerical value of λ^* to compute the optimal solution(s) to the primal problem (P).

4. Let $f : D \to \mathbb{R}$ be a *convex* function on a convex set $D \subseteq \mathbb{R}^n$.

(a) Show that any local minimizer $x^* \in D$ of f on D is in fact a global minimizer of f on D. **You may not assume f is differentiable!**

(b) Assume that f is differentiable. Prove that x^* is a global minimizer of f on D if and only if $\nabla f(x^*)^T(x - x^*) \geq 0$ for all $x \in D$.

(c) Assume that f is differentiable and D is open. Prove that x^* is a global minimizer of f on D if and only if $\nabla f(x^*) = 0$.

(d) Assume that f is twice continuously differentiable and D is open. Prove that the Hessian $\nabla^2 f(x)$ is a symmetric, positive semi–definite matrix at all points $x \in D$.