COMPREHENSIVE EXAMINATION

Math 650 – Optimization

August 1998

You must show all your work for full credit!

INSTRUCTIONS:
You must do problem 1 (35 points). Do either 2 or 3 (30 points), and either 4 or 5 (35 points), for a total of 3 problems and 100 points. You may do either 6 or 7 for extra credit (5 points).

Q1. Consider the optimization problem

$$\begin{array}{rl} \min & r \\ \text{subject to} & x_1^2 + x_2^2 - r \le 0 \\ & (x_1 - 3)^2 + x_2^2 - r \le 0 \\ & (x_1 - 2)^2 + (x_2 - 1)^2 - r \le 0. \end{array}$$

(a) Show that the above problem is exactly the problem of finding the smallest circle that contains the points $(0, 0)$, $(3, 0)$, and $(2, 1)$ in the plane (or the triangle determined by these points). Obviously r is the square of the radius of the circle. What do the variables (x_1, x_2) correspond to?

(b) Write the Lagrangian for the problem using the multiplier λ_i for the ith constraint, $i = 1, 2, 3$. Explane why λ_0 can be assumed to be 1 in the Fritz John conditions. Then write down the KKT conditions, including the complementarity conditions. Use the KKT conditions to verify that λ_i's add up to one, express x_1, x_2 in terms of the multipliers λ_i. Finally, show that the optimal x_i^*'s are non–negative.

(c) Show that at least one multiplier must be zero by demonstrating that assumption that they are all non–zero leads to the contradiction that one of the variables $x_i^* < 0$.

(d) Show that assumptions $\lambda_2 = 0$, and $\lambda_1 \neq 0, \lambda_3 \neq 0$ lead to contradiction.

(e) Similarly, one can show that the assumptions $\lambda_1 = 0$, $\lambda_2 \neq 0, \lambda_3 \neq 0$ lead to a contradiction (do not show this). Use all the information so far to find the optimal solution(s) $(x_1^*, x_2^*, r^*, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ to the optimization problem.

(f) Give a theoretical reason as to why the optimal solution you found above is the global one(s).

Q2. Let $C \subseteq \mathbb{R}^n$ be a closed, convex set and $x \in \mathbb{R}^n$ a point such that $x \notin C$. Recall that the projection $\pi_C(x)$ is the (unique) point in C that is closest to x, i.e., $||x - \pi_C(x)|| \leq ||x - z||$ for all $z \in C$. Also recall that the distance from x to C is $d_C(x) := ||x - \pi_C(x)||$.

(a) Write the *variational inequality* which gives a characterization of $\pi_C(x)$.

(b) Show that if $D \supseteq C$ is a closed convex set, then $d_D(x) \leq d_C(x)$. Conclude that if the halfspace

$$H^-_{d,\alpha} := \{z \in \mathbb{R}^n : \langle d, z\rangle \leq \alpha\}, \quad d \neq 0,$$

contains C, then $d_{H^-_{d,\alpha}}(x) \leq d_C(x)$.

(c) Show that the variational inequality in (a) implies

$$(i)\ C \subseteq H^-_{d,\alpha}, \qquad (ii)\ d_C(x) = d_{H^-_{d,\alpha}}(x),$$

where $d = x - \pi_C(x)$ and $\alpha = \langle d, \pi_C(x)\rangle$.

(d) Show that all these lead to the following, geometrically appealing "duality" result: *the shortest distance from a point x to a convex set C not containing x is equal to the maximum among the distances from x to half spaces containing C.*

Q3. Recall that the polar of a convex set $C \subseteq \mathbb{R}^n$ is the set

$$C^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 1, \quad \forall x \in C\}.$$

(a) Show that if C is a convex body (a compact convex set) containing zero in its interior ($0 \in C^0$), then so is C^*.

(b) Show that if C is convex body containing zero in its interior, then $C^{**} \subseteq C$. (Since $C \subseteq C^{**}$ (do not show this), this implies $C^{**} = C$.) *Hint:* use a separation argument.

(c) Show that if $P := \{x : Ax \leq b\}$ is a polyhedron, then P^* is also one, and describe P^*. *Hint:* use affine Farkas Lemma.

Q4. In the quadratic program

$$\min\{\frac{1}{2}x^TQx + c^Tx : Ax \leq b\},$$

Q is an $n \times n$ symmetric, positive definite matrix. Write the Lagrangian function and use it to determine the dual program. In particular, show that the dual program can be written in the form

$$\max\{\frac{1}{2}y^TRy + d^Ty : y \geq 0\},$$

where R is an $n \times n$ symmetric matrix and $d \in \mathbb{R}^n$.

Q5. Solve, that is, find the optimal solution(s) to the constrained minimization problem

$$\begin{array}{rl} \min & \dfrac{x_1}{x_2} + \dfrac{x_2}{x_3} + \dfrac{x_3}{x_1} \\ \text{subject to} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

(a) By forming the Lagrangian and setting up the Karush–Kuhn–Tucker (KKT) conditions,

(b) By making the substitutions $x_i = e^{t_i}$ and then solving the unconstrained minimization problem in the variables t_1, t_2, t_3,

(c) By applying the arithmetic–geometric mean (AGM) inequality to the objective function in either (a) or (b). State the AGM inequality first!

(d) The original problem is not a convex program. Yet, the optimal solution(s) you found is (are) global one(s). Why is this so, give a reason.

Q6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. It is well–known that f is a convex function if and only if

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \qquad \forall x, y \in \mathbb{R}^n. \tag{1}$$

Show that (1) is equivalent to

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n. \tag{2}$$

Hint: To prove (2) implies (1), use the mean value theorem to f, then apply (2).

Q7. Let $P := \{x : Ax \leq b\}$ be a polytope (bounded polyhedron) where A is an $m \times n$ matrix. Show that the rows $\{a_1, \ldots, a_m\}$ of A span $\mathbb{R}^n$. *Hint:* pick a basis $\{c_1, \ldots, c_n\}$ for $\mathbb{R}^n$. Formulate the duals of the linear programs $\{\max c_i^T x : Ax \leq b\}$ and invoke the LP duality.