COMPREHENSIVE EXAMINATION

Optimization

January 1995

♢ **You must show all your work for full credit.** ♢

INSTRUCTIONS:

Do problems 1, 4, 5, and either 2 or 3, for a total of 4 problems.

Q1. Consider the linear program

$$\begin{aligned} \min \quad & 6X_1 + 8X_2 + 16X_3 \\ \text{subject to} \quad & 2X_1 + X_2 \geq 5 \\ & X_2 + 2X_3 \geq 4 \\ & X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0. \end{aligned}$$

(a) Determine (write explicitly) the dual linear program by using the Lagrangian method.

(b) Find a basic solution of the above LP by using the big–M method. (Bring into the basis the variable which makes the largest improvement per unit).

(c) Solve the linear program in (b) using the simplex method. (Again, bring into the basis the variable which makes the largest improvement per unit). What is the solution?

(d) Using the results of (c), determine the optimal solution to the dual linear program you found in (a).

Q2. Consider the convex optimization problem

$$\begin{aligned} \min \quad & \frac{1}{2}X_1^2 - X_2 \\ \text{subject to} \quad & X_1 + X_2 \leq 5 \\ & 3X_1 - X_2 \leq 2 \end{aligned}$$

(a) Find explicitly the dual program to the above primal problem.

(b) Solve the dual program.

(c) Use the solution(s) to the dual program to calculate the solution(s) to the primal program. Interpret the the dual solution in terms of perturbation of the primal problem.

Q3. Consider the optimization problem

$$\begin{aligned} \max \quad & X_1^2 - X_2 \\ \text{subject to} \quad & X_1^2 + X_2^2 \leq 9 \\ & 3X_1 - X_2 \leq 6 \end{aligned}$$

(a) Write the Lagrangian for the problem, and the KKT conditions.

(b) Find the point(s) which satisfy the KKT (Karush–Kuhn–Tucker) conditions. (IGNORE the KKT points at which both constraints are active).

(c) Use the second order test to determine which of the KKT point(s) are local maximum points.

Q4. Consider the optimization problem

$$\begin{array}{rl} min & xyz \\ \text{subject to} & x + y + z = 0, \\ & x^2 + y^2 + z^2 = 1. \end{array}$$

(a) Form the Lagrangian function $L(x, y, z, \lambda, \mu) = xyz + \lambda(x + y + z) + \mu(x^2 + y^2 + z^2 - 1)$, where λ and μ are the multipliers. Write the KKT conditions for the problem.

(b) Use the KKT conditions to show that $3xyz = -2\mu$, and argue that we must have $\mu > 0$. *Hint:* We are minimizing xyz.

(c) Use the KKT conditions to show that

$$x(\lambda + 2\mu x) = y(\lambda + 2\mu y) = z(\lambda + 2\mu z) = -xyz = \frac{2\mu}{3},$$

and argue that x, y, z must be the roots of the equation

$$u^2 + \gamma u - \frac{1}{3} = 0, \tag{1}$$

where $\gamma = \frac{\lambda}{2\mu}$.

(d) Using (b) and the KKT conditions, argue that if (x^*, y^*, z^*) is an optimal solution with $x^* \leq y^* \leq z^*$, then $x^* < 0 < y^* \leq z^*$. Then use (c) to show that $y^* = z^*$.

(e) Using (1) argue that $y^* = z^* = -(x^* + y^*) = \gamma$, and that $x^* y^* = -1/3$ so that $x^* = \frac{-1}{3\gamma}$.

(f) Use $x^* + y^* + z^* = 0$ to show that $y^* = z^* = \gamma = \frac{1}{\sqrt{6}}$, and $x^* = \frac{-2}{\sqrt{6}}$. You have now solved a special case of a problem which occurs in Karmarkar's projective algorithm for linear programming.

5. Consider the following quadratic program (QP)

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx + c^x \\ \text{subject to} & Ax \geq b, \end{array}$$

where $Q \in R^{n\times n}$ is a symmetric positive definite matrix, $A \in R^{m\times n}$, $x \in R^n$, $c \in R^n$, and $b \in R^m$.

(a) Argue that there exits a unique solution to (QP), say x^*. Write the KKT conditions which must be satisfied at x^*.

(b) State the variational inequality (VI) for a general convex program $\min\{f(x) : x \in C\}$ where $C \subseteq R^n$ is a closed convex set and f is differentiable. Write the particular (VI) which must be satisfied at the above x^*.

(c) One version of Farkas Lemma states:

`Suppose that any` u `satisfying` $Gu \leq g$ `also satisfies` $h^T u \leq \alpha$.
`Then there exists` $\lambda \geq 0$ `such that` $G^T\lambda = h$ `and` $g^T\lambda \leq \alpha$.

Show that applying Farkas Lemma to the (VI) in (b) gives the existence of $\lambda^* \geq 0$ satisfying

$$Qx^* + c = A^T\lambda^*, \qquad b^T\lambda^* \geq (Qx^* + c)^T x^*.$$

Argue that we actually have $b^T\lambda^* = (Qx^* + c)^T x^*$. Compare these conditions with the KKT conditions obtained in (a).

(d) Using the Lagrangian approach, write explicitly the dual to the quadratic program (QP). *Hint:* Use the fact that Q is positive definite.