Comprehensive Examination

OPTIMIZATION

August 1994

INSTRUCTIONS:

Do problem 1, either 2 or 3, and either 4 or 5, for a total of 3 questions.

1. (36 points) Consider the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2 \\ & 2x_1 - x_2 \leq 6 \\ & x_1 + 3x_2 \leq 10 \\ & x_1 \geq, \ x_2 \geq 0 \end{aligned}$$

(a) Write the Lagrangian function of the above linear program. Using the Lagrangian function, determine (write explicitly) the dual linear program.

(b) Convert the linear program displayed above into a minimization linear program in standard equality form. (This is the form in which the only linear inequalities are the non–negativity constraints on the variables.)

(c) Solve the linear program in (b) using the simplex method.

(d) Using the results of (c), determine the optimal solution to the dual linear program you found in (b).

2. (32 points) Consider the optimization problem

$$\begin{aligned} & \max x_1^2 + x_2^2 - 4x_1 + 4 \\ & \text{s.t. } x_1 - x_2 + 2 \geq 0 \\ & \quad -x_1^2 + x_2 - 1 \geq 0 \\ & \quad x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

(a) Solve the problem geometrically, by sketching it.

(b) Show that the point (x_1^*, x_2^*) you found in (a) satisfies the KKT (Karush–Kuhn–Tucker) conditions.

(c) Use the second order test to verify that (x^*, y^*) is a local maximum. (It is in fact the global maximum point.)

3. (32 points) Consider the optimization problem

$$\begin{aligned} \max\ & x_1^2 + x_2 \\ \text{s.t.}\ & x_1^2 + x_2^2 - 9 \leq 0 \\ & x_1 + x_2 - 1 \leq 0 \end{aligned}$$

(a) Sketch the feasible region and the level curves of the objective function.

(b) Determine which of the following four points satisfy the KKT (Karush–Kuhn–Tucker) conditions:

$$(i)\ \ (x_1^*, x_2^*) = (\frac{-\sqrt{35}}{2}, \frac{1}{2}), \quad (ii)\ \ (x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{2}),$$

$$(iii)\ \ (x_1^*, x_2^*) = (\frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}), \quad (iv)\ \ (x_1^*, x_2^*) = (\frac{1-\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}).$$

(c) Use the second order test to determine which of the four points in (b) are local maximum points.

4. (32 points) Consider the optimization problem

$$\begin{aligned} \min\ & \sum_{i=1}^{n} \frac{1}{x_i} \\ \text{s.t.}\ & \prod_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

(a) Determine the point satisfying the KKT (Karush–Kuhn–Tucker) conditions.

(b) Show that the KKT point found in (a) satisfies the second order necessary conditions. (Hint: it may help to verify the second order conditions for n=2, or n=3 first, in order to see the pattern of the proof in the general case.)

(c) Use the results of (a) and (b) to prove the harmonic–geometric mean inequality

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \leq (\prod_{i=1}^{n} x_i)^{1/n}.$$

5. (32 points) Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$, we have

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y).$$

(a) Prove that f is convex if and only if for any $x, y \in \mathbb{R}^n$,

$$f(y) \ge f(x) + \nabla f(x)^T (y - x).$$

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $C \subseteq \mathbb{R}^n$ be a closed convex set. Show that a point $x^* \in C$ minimizes f on C if and only if the following *variational inequality* holds true:

$$\nabla f(x^*)^T (x - x^*) \ge 0, \quad \text{for all } x \in C.$$