Comprehensive Examination

OPTIMIZATION

August, 1993

INSTRUCTIONS:

Do problem 1, either 2 or 3, and either 4 or 5, for a total of 3 questions.

1. (36 points) Consider the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2 \\ & 2x_1 - x_2 \leq 6 \\ & x_1 + 3x_2 \leq 10 \\ & x_1 \geq, \ x_2 \geq 0 \end{aligned}$$

(a) Write the Lagrangian function of the above linear program. Using the Lagrangian function, determine (write explicitly) the dual linear program.

(b) Convert the linear program displayed above into a minimization linear program in standard format.

(c) Solve the linear program in (b) using the simplex method.

(d) Using the results of (c), determine the optimal solution to the dual linear program you found in (b).

2. (32 points) Consider the optimization problem

$$\begin{aligned} \max \ & x_1^2 + x_2^2 - 4x_1 + 4 \\ \text{s.t.} \ & x_1 - x_2 + 2 \geq 0 \\ & -x_1^2 + x_2 - 1 \geq 0 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

(a) Solve the problem geometrically.

(b) Show that the point (x_1^*, x_2^*) you found in (a) satisfies the KKT (Karush–Kuhn–Tucker) conditions.

(c) Use the second order test to verify that (x^*, y^*) is a local maximum. (It is in fact the global maximum point.)

3. (32 points) Consider the optimization problem

$$\begin{aligned} \max\ & x_1^2 + x_2 \\ \text{s.t. } & x_1^2 + x_2^2 - 9 \leq 0 \\ & x_1 + x_2 - 1 \leq 0 \end{aligned}$$

(a) Sketch the feasible region and the level curves of the objective function.

(b) Determine which of the following four points satisfy the KKT (Karush–Kuhn–Tucker) conditions:

$$(i)\ \ (x_1^*, x_2^*) = (\frac{-\sqrt{35}}{2}, \frac{1}{2}), \quad (ii)\ \ (x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{2}),$$

$$(iii)\ \ (x_1^*, x_2^*) = (\frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}), \quad (iv)\ \ (x_1^*, x_2^*) = (\frac{1-\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}).$$

(c) Use the second order test to determine which of the four points in (b) are local maximum points.

4. (32 points) Consider the optimization problem

$$\begin{aligned} \min\ & \sum_{i=1}^{n} \frac{1}{x_i} \\ \text{s.t. } & \prod_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

(a) Determine the point satisfying the KKT (Karush–Kuhn–Tucker) conditions.

(b) Show that the KKT point found in (a) satisfies the second order necessary conditions. (Hint: it may help to verify the second order test for n=2, or n=3 first, in order to see the pattern of the proof in the general case.)

(c) Use the results of (a) and (b) to prove the harmonic–geometric mean inequality

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \leq (\prod_{i=1}^{n} x_i)^{1/n}.$$

5. **(32 points)** Let $S_n = \{\lambda : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\}$ be the unit simplex in E_n. Consider the function

$$\Theta(x, \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x),$$

where $f_i : E_n \to E$ is a differentiable convex function, $i = 1, \ldots, m$. Suppose that the point (x^*, λ^*), $\lambda^* \in S_n$, is a saddle point for Θ, that is,

$$\Theta(x^*, \lambda) \leq \Theta(x^*, \lambda^*) \leq \Theta(x, \lambda^*), \quad \forall x \in E_n,\ \lambda \in S_n.$$

(a) Show that the point x^* minimizes the so–called maximum function, which is the (convex) function given by $f(x) = \max\{f_i(x), \ldots, f_m(x)\}$.

(b) Let $I(x^*) = \{i : f_i(x^*) = f(x^*)\}$. Show that the following conditions are satisfied:

$$\lambda^* \geq 0,\ \lambda_i^* = 0\ \forall i \notin I(x^*),$$
$$\sum_{i=1}^{m} \lambda_i = 1, \quad \sum_{i=1}^{m} \lambda_i^* f_i'(x^*) = 0.$$