MASTER'S COMPREHENSIVE EXAM IN
Math 603 -MATRIX ANALYSIS
January 2016

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

Q1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that (i) $f(x) \geq 0$ for all $x \in \mathbb{R}^n$; (ii) for any $\lambda \in \mathbb{R}$, $f(\lambda x) = |\lambda| f(x)$ for all $x \in \mathbb{R}^n$; and (iii) $f(x+y) \leq f(x) + f(y)$ for any $x, y \in \mathbb{R}^n$. (Such an f is called a seminorm on $\mathbb{R}^n$.) Let W be the zero set of f, i.e., $W := \{x \in \mathbb{R}^n \mid f(x) = 0\}$.

(a) Show that W is a subspace of $\mathbb{R}^n$.

(b) Show that f is a norm on $\mathbb{R}^n$ if and only if $W = \{0\}$.

(c) Suppose W is a proper subspace of $\mathbb{R}^n$, and let $W^\perp$ be the orthogonal complement of W. Show that for any $x \in \mathbb{R}^n$, there exists a unique $u_x \in W^\perp$ such that $f(x) = f(u_x)$. Furthermore, show that $\{u \in W^\perp \mid f(u) = 0\} = \{0\}$.

Q2 Solve the following problems.

(a) Let P be a symmetric positive semidefinite matrix such that $P^{100} = P^{20}$. Show that $\text{rank}(P) = \text{trace}(P)$.

(b) Let P be an $n \times n$ symmetric real matrix, and $c \in \mathbb{R}^n$. Suppose that $\{x^T P x + c^T x \mid x \in \mathbb{R}^n\}$ is bounded below. Show that P is positive semidefinite, and c is in the range of P.

Q3 Let A be an $n \times n$ real matrix, prove that

(a) A is skew-symmetric if and only if $A^2 = -AA^T$.

(b) The matrix e^A is orthogonal if A is skew-symmetric.

Q4 Prove the following two statements:

(a) For matrices A, B and C such that AB, BC and ABC are all well-defined, prove that $rank(AB) + rank(BC) \leq rank(B) + rank(ABC)$.

(b) Let A be an $n \times n$ matrix. Suppose that there exists a natural number N such that

$$rank(A^N) = rank(A^{N+1}),$$

prove that

$$rank(A^N) = rank(A^{N+1}) = rank(A^{N+2}) = rank(A^{N+3}) = \cdots .$$

Q5 Consider the vector space $R^{n \times n}$ of all real $n \times n$ matrices.

(a) By describing a basis, find the dimension of $R^{n \times n}$.

(b) Given any $A \in R^{n \times n}$, show that $S(A) := span\{I_n, A, A^2, A^3, \ldots\}$ is a subspace of $R^{n \times n}$. When $n > 1$, can this subspace be equal to $R^{n \times n}$?

(c) If A is invertible, show that $S(A^{-1}) = S(A)$.