Comprehensive Examination

Statistics 651 - Basic Probability

100 points

January 20, 2016

Name: ______________________________

- You **have** 150 minutes
- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer

Q1. There are 6 children in the kindergarten. Each child can distinguish between right shoe and left shoe. In the way home each child randomly chooses one left shoe and independently randomly chooses one right shoe. What is the probability that each child will come home with a wrong pair (i.e., both his/her left and right are wrong) ?

Q2. One student wrote an algorithm to sort the set S of 5 distinct values, i.e., to put these values in increasing order. Step 1: choose one value x at random from S, and compare each of the other 4 values with x, noting which are smaller and which are larger than x. Denote S_x the set of elements smaller than x and $\overline{S}_x$ the set of elements larger than x. Step 2: perform separately Step 1 on S_x and on $\overline{S}_x$. Step 3: continue with Steps 1 and 2 until you sort all 5 distinct values. Find expected number of comparisons needed by this algorithm to sort all 5 distinct values.

Q3. Let $X_1, X_2, \ldots$ be a sequence of independent $\exp(1)$ random variables. Define the random variable N by $N = \min\{n : X_n > 1\}$.

(a) Calculate the probability $P(X_N > 2 \mid N = 4)$.

(b) Find $E(X_N)$.

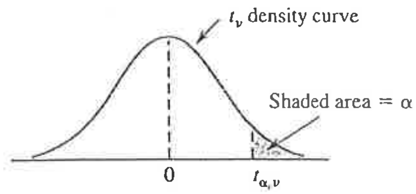
Q4. (a) Suppose Z is standard normal random variable. Denote $Y = e^X$. Find $Var(Y)$.

(b) Suppose that the random variables $X_1, \ldots, X_{12}, X_{13}$ are independent $N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$, $\sigma^2 > 0$ unknown parameters. Denote $S_{12}^2 = \sum_{i=1}^{12} (X_i - \overline{X}_{12})^2$, $\overline{X}_{12} = \frac{1}{12}\sum_{i=1}^{12} X_i$, and $Y_{13} = \dfrac{X_{13} - \overline{X}_{12}}{S_{12}}$.

Find the value a, that such $P(-a \le Y_{13} \le a) = 0.90$ (you will need the attached table).

Q5. A needle of length 1 inch dropped randomly on a plane ruled with parallel lines 2 inches apart. What is the probability that the needle will cross a line ? You have to present all assumptions that you made along with the calculations.

Table A.5 Critical Values for t Distributions



ν \ α	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
32	1.309	1.694	2.037	2.449	2.738	3.365	3.622
34	1.307	1.691	2.032	2.441	2.728	3.348	3.601
36	1.306	1.688	2.028	2.434	2.719	3.333	3.582
38	1.304	1.686	2.024	2.429	2.712	3.319	3.566
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	1.299	1.676	2.009	2.403	2.678	3.262	3.496
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Comprehensive Examination

Statistics 651 - Basic Probability

100 points

January 21 , 2015

Name: ______________________________

- You **have** 150 minutes
- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer

Q1. $X_1, X_2, \ldots, X_n$ are independent Poisson variables with means $\lambda_1, \ldots, \lambda_n$, respectively. Compute for $r < n$ and $t \leq u$, $P(X_1 + X_2 + \ldots + X_r = t \,|\, X_1 + \ldots + X_n = u)$. What happens if $\lambda_1 = \ldots = \lambda_n$?

Q2. An urn contains 10 balls numbered $1, 2, \ldots, 10$. We select 5 balls at random without replacement. Let M be the maximum number in the sample drawn.

(a) Find the probability mass function of M.

(b) Find $E(M)$.

Q3. The random variables U and V each takes values $+1$, -1. Their joint distribution is given by

$$P(U = 1) = P(U = -1) = \frac{1}{2}, \quad P(V = 1 \,|\, U = 1) = \frac{1}{3} = P(V = -1 \,|\, U = -1),$$
$$P(V = -1 \,|\, U = 1) = \frac{2}{3} = P(V = 1 \,|\, U = -1).$$

(a) Find the probability that $x^2 + Ux + V = 0$ has at least one real root.

(b) Find the expected value of the larger root, given that there is at least one real root.

Q4. The double exponential distribution has density function $f_X(x) = \frac{1}{2} c e^{-c|x|}$ for $x \in \mathbb{R}$, where $c > 0$ is a parameter of the distribution. Find $Var\,(X)$.

Q5. Let X and Y have the joint probability density function

$$f_{X,Y}\,(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}}, \quad -\infty < x, y < \infty,$$

where ρ is a constant satisfying $-1 < \rho < 1$.

(a) Find the joint probability density function of $U = X + Y$ and $V = X - Y$.

(b) Is U independent of V ?

(c) Find $E\,(Y \,|\, X = 1)$.

Comprehensive Examination

Statistics 651 - Basic Probability.

100 points

January 22, 2014

Name: ______________________________

- You have 120 minutes.
- You can use only calculator, but **not cell phone**
- You must show all proof details/calculations which lead to the answer.

Q1. A man possesses **three coins**, one of which is double-headed, one is double-tailed, and one is normal. He shuts his eyes, picks one at random and tosses it. He opens his eyes and sees that the coin is showing tail. **He discards this coin**, picks another at random, and tosses it. What is the probability that it shows head ?

Q2. A secretary types 5 different letters together with matching envelopes, drops the pile down the stairs, and then places the letters **randomly** in the envelopes. Find the probability that

(a) there is **no** match.

(b) exactly **two** matches.

Q3. Let X, Y be independent $N(0,1)$ random variables. Set $m = \min(X,Y)$, $M = \max(X,Y)$. Find correlation between m and M.

Hint: You can use the following trivial identities:

$mM = XY,\ M + m = X + Y,\ M - m = |X - Y|$.

Q4. Let Y and $\{U_r : r \geq 1\}$ be independent random variables, where

$P(Y = y) = \dfrac{1}{(e-1)y!}$, for $y = 1, 2, 3, \ldots$ and $U_r, r \geq 1$ are continuous uniform on $[0,\ 1]$.

Let $M = \max(U_1, U_2, \ldots, U_Y)$. Find

(a) $E(M)$.

(b) $Var(M)$.

Hint: $\dfrac{x}{x+1} = 1 - \dfrac{1}{x+1}$, $\dfrac{x}{x+2} = 1 - \dfrac{2}{x+2}$.

Q5. Let X and Y have the joint probability density function

$f_{X,Y}(x,\ y) = kx(y-x)e^{-y},\ 0 < x \leq y < \infty.$

(a) find k.

(b) Determine the nature of regression of X on Y.

Comprehensive Examination

Statistics 651 - Basic Probability.

100 points

August 19, 2014

Name: ______________________________

- You have 150 minutes.
- You can use only calculator, but not cell phone
- You must show all proof details/calculations which lead to the answer.

Q1. The number of coins that Josh spots on the gound when walking to work is a Poisson random variable with mean 6. Each coin is equally likely to be a penny(1¢), a nickel(5¢), a dime(10¢), or a quarter(25¢). Josh ignores the pennies but picks up the other coins.

(a) Find the expected amount of money that Josh picks up on his way to work.

(b) Find the variance of the amount of money that Josh picks up on his way to work.

Q2. In a party of 7 married couples everybody dances. Every gentleman dances with every one of the ladies with the same probability.

(a) What is the probability that no gentleman dances with his own wife ?

(b) What is the probability that at least two gentlemen dance with their own wives ?

Q3. Let X and Y have the joint probability density function $f_{X,Y}(x,y) = \begin{cases} Ke^{-x-y}, & 0 < y < x < \infty \\ 0, & otherwise. \end{cases}$

(a) Find the probability density function of $\frac{X}{X+Y}$.

(b) Find $cov(X,Y)$.

Q4. Let $X_1, X_2, \ldots$ be a sequence of independent continuous uniform $(0,1)$ random variables. Define the random variable N by $N = \min\{n : X_n > 2/3\}$.

(a) Calculate the probabilities $P(X_N > 3/4 \mid N = 3)$ and $P(X_N > 3/4 \mid N = 4)$.

(b) Is N independent of X_N ? Prove or disprove with your arguments.

Comprehensive Examination

Statistics 653 - Basic Mathematical Statistics

100 points

January 20 , 2016

Name: ______________________________

- You **have** 150 minutes
- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer

Q1. A circle **C** is drawn inside of a square sheet of paper. The length of the paper side is known precisely and equals 10 inches. Alice and Bob want to estimate the area θ of the circle and suggest two different approaches.

(A) Alice approach: she suggests to generate independent random vectors $(X_1, Y_1), \ldots, (X_{13}, Y_{13})$, where X_i and Y_i are **iid** random variables with $U[0, 10]$ distribution, $i = 1, \ldots, 13$. Then estimate **A** based on iid random variables $Z_1, \ldots, Z_{13}$, where

$$Z_i = \mathbb{1}_{((X_i,Y_i)\in \mathbf{C})},\ i = 1, \ldots, 13.$$

(i) Specify a statistical model based on $Z_1, \ldots, Z_{13}$ and parameterized by θ (the area of the circle) (do not forget to describe the parameter space).

Is the parametrization of this model identifiable ?

(ii) Find MLE $\widehat{\theta}_A$ of θ based on $(Z_1, \ldots, Z_{13})$ and show that it is unbiased for θ.

(iii) Is $\widehat{\theta}_A$ UMVUE? You have to justify all steps which lead to the answer!

(B) Bob approach: suggests to measure the diameter D of the circle with a ruler. Since his measurement is noisy, he repeats it 13 times and assumes that his data $V_1, \ldots, V_{13}$ is a sample from $N(D, 1)$.

(iv) Find the estimator of θ based on method of moments. If it is not an unbiased estimator, then "correct" it in order to get an unbiased estimator. Call the final estimator $\widehat{\theta}_B = \widehat{\theta}_B\,(V_1, \ldots, V_{13})$.

(v) Calculate MSE of the two estimators $\widehat{\theta}_A$ and $\widehat{\theta}_B$ and identify the set/sets of the θ-values such that $MSE(\widehat{\theta}_A) < MSE(\widehat{\theta}_B)$.

Q2. Consider independent random variables X_1, X_2, with $X_i \sim N(\theta_i, 1)$, $i = 1, 2$.

(i) Find the most powerful (MP) test of size $\alpha = 0.10$ for the problem:

$$\begin{aligned} H_0 &: \quad (\theta_1, \theta_2) = (0, 0) \\ H_1 &: \quad (\theta_1, \theta_2) = (1, 0). \end{aligned}$$

(ii) Find the size $\alpha = 0.10$ GLRT for the problem:

$$\begin{aligned} H_0 &: \quad (\theta_1, \theta_2) = (0, 0) \\ H_1 &: \quad (\theta_1, \theta_2) = (1, 0) \text{ or } (\theta_1, \theta_2) = (0, 1). \end{aligned}$$

(iii) Calculate the power of MP test.

Calculate the power of GLRT when $(\theta_1, \theta_2) = (0, 1)$.

Q3. Let X_1, X_2, X_3 be independent identical distributed random variables from the continuous uniform distribution with support $\left[\frac{1}{2}\theta, \frac{3}{2}\theta\right]$, $\theta > 0$. The goal is to construct CI confidence interval (CI) for the parameter θ at level $1 - \alpha = 0.9$ while one of the CI bounds is given.

(i) Find the pivot based on $X_{(1)} = \min(X_1, X_2, X_3)$ and find the the upper bound $U = U(X_{(1)})$ for the CI $\left[\frac{2}{3}X_{(1)}, U\right]$. Calculate the expected length of this CI.

(ii) Find the pivot based on $X_{(3)} = \max(X_1, X_2, X_3)$ and find the the lower bound $L = L(X_{(3)})$ for the CI $\left[L, 2X_{(3)}\right]$. Calculate the expected length of this CI.

Probability Theory,
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TABLE I. Area Φ (z) under the normal density to the left of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Auxiliary table of quantiles of the normal d.f.

$1-\alpha$	$z(1-\alpha)$	$1-\alpha$	$z(1-\alpha)$	$1-\alpha$	$z(1-\alpha)$
.50	0	.91	1.341	.995	2.576
.55	.126	.92	1.405	.999	3.090
.60	.253	.93	1.476	.9995	3.291
.65	.385	.94	1.555	.9999	3.719
.70	.524	.95	1.645	.99995	3.891
.75	.674	.96	1.751	.99999	4.265
.80	.842	.97	1.881	.999995	4.417
.85	1.036	.98	2.054	.999999	4.753
.90	1.282	.99	2.326	.9999999	5.199

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January 21 , 2015

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- You **have** 150 minutes
- You **can use only** calculator, but **not cell phone**
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Q1. Let $Z_1, Z_2, \ldots$ be i.i.d. *Ber* (θ) , $\theta \in (0,1)$. We know that there is no unbiased estimator of the odds ratio $\theta/(1-\theta)$, based on the fixed size sample of Z's. Let

$$X_1 = \min\{n : Z_n = 0\},$$

be the number of tosses till first 0 occurs.

(i) Find MLE of θ on the basis of X_1. Find the MLE of $\eta(\theta) = \dfrac{\theta}{1-\theta}$.

(ii) Is MLE $\widehat{\eta}(X_1)$ unbiased estimator of η ? Is it UMVUE ?

(iii) Does the variance of MLE $\widehat{\eta}(X_1)$ attain Cramer-Rao lower bound bound ?

(iv) Encouraged by his progress, the statistician suggests to count the tosses till m, $m \geq 1$ zeros occur:

$$X_m = \min\left\{n : \sum_{i=1}^{n}(1 - Z_i) = m\right\}.$$

Find MLE of $\eta(\theta) = \dfrac{\theta}{1-\theta}$ on the basis of X_m.

Q2. Let $X_1, \ldots, X_n$ be a random sample from the population with the probability density function

$$f(x;\,\theta) = \begin{cases} c(\theta)xe^{\theta x^2/2} & 0 \leq x \leq 1 \\ 0 & \textit{otherwise}, \end{cases}$$

where $c(\theta)$ is normalized constant (which is function of θ), and $\theta > 0$ is a parameter.

(i) Find the most powerful (MP) test for the problem:

$$H_0 : \quad \theta = 0$$
$$H_1 : \quad \theta = 2.$$

You do not need to specify the critical value as a function of the test size.

(ii) Assume that in the sample of size $n = 1$, $X_1 = 0.9$. Calculate the $p-value$ value of the MP test in (i). What is your decision if $\alpha = 0.10$?

(iii) Is MP test that you found in (i) also UMP test for $\begin{matrix} H_0: & \theta \leq 0 \\ H_1: & \theta > 0 \end{matrix}$? You have to show all steps which lead to the answer!

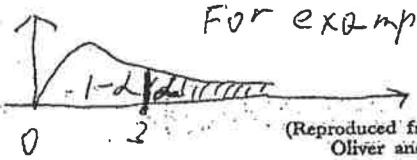
Q3. Let $X_1, \ldots, X_{10}$ be independent identical distributed random variables from the exponential distribution with the mean $E(X_1) = \lambda$, $\lambda > 0$. The goal is to construct the confidence interval (CI) for the parameter λ at level $1 - \alpha = 0.9$. The lower bound of this CI is known for us, because $\lambda > 0$ and therefore we only need to construct upper bound, i.e. one-sided CI.

(i) Find the pivot based on minimal sufficient statistic $\sum_{i=1}^{10} X_i$ and construct one-sided CI.

(ii) Find the pivot based on $X_{(1)} = \min(X_1, \ldots, X_{10})$ and construct one-sided CI.

(iii) Which one of above confidence intervals has smallest expected length ? You will need here the attached table.

For example if $\vartheta=5$ and $\alpha=0.7$ then $\chi^2_{5,0.7}=3$



Appendix Table 3.–Significance points of χ^2

(Reproduced from Table III of Sir Ronald Fisher's *Statistical Methods for Research Workers*, Oliver and Boyd Ltd., Edinburgh, by kind permission of the author and publishers)

ν	P = 0.99	0.98	0.95	0.90	0.80	0.70	0.50	0.30	0.20	0.10	0.05	0.02	0.01
1	0.0^3157	0.0^3628	0.0^2393	0.0158	0.0642	0.148	0.455	1.074	1.642	2.706	3.841	5.412	6.635
2	0.0201	0.0404	0.103	0.211	0.446	0.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.345
4	0.297	0.429	0.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277
5	0.554	0.752	1.145	1.160	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086
6	0.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666
10	2.358	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.821	18.549	21.026	24.054	26.217
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.204	30.144	33.687	36.191
20	8.260	9.237	10.851	12.443	14.578	16.266	19.337	22.775	25.038	28.412	31.410	35.020	37.566
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	30.113	44.140	46.963
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588
30	14.953	16.306	18.493	20.599	23.364	25.508	29.336	33.530	36.250	40.256	43.773	47.962	50.892

Note—For values of ν greater than 30 the quantity $\sqrt{(2\chi^2)}$ may be taken to be distributed normally about mean $\sqrt{(2\nu-1)}$ with unit variance.

Comprehensive Examination

Statistics 653 - Basic Mathematical Statistics.

100 points

August 19, 2014

Name: ______________________________

- You **have** 150 minutes.
- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer.

Q1. Consider the problem of estimating the unknown parameter $\theta \in \mathbb{R}$ from the sample of n independent pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$, $n \geq 3$, where

$$Y_i = \theta X_i + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

where *covariates* X_i's and the *noise* ε_i's are i.i.d. random variables with $N(0,1)$ distribution.

(i) Show that the likelihood function is given by

$$L(X, Y; \theta) = \left(\frac{1}{2\pi}\right)^n e^{-\frac{1}{2}\sum_{i=1}^n X_i^2 - \frac{1}{2}\sum_{i=1}^n (Y_i - \theta X_i)^2}.$$

(ii) Find the minimal sufficient statistic for this model.

(iii) Show that the minimal sufficient statistic is incomplete.

(iv) Find MLE of θ. Is MLE unbiased estimator of θ ?

(v) Does the variance of MLE attain Cramer-Rao bound ?

Q2. Consider independent random variables X_1, X_2, with $X_i \sim N(\theta_i, 1)$, $i = 1, 2$.

(i) Find the most powerful test of size α for the problem:

$$H_0 : \quad (\theta_1, \theta_2) = (0, 0)$$
$$H_1 : \quad (\theta_1, \theta_2) = (1, 0).$$

(ii) For the size $\alpha = 0.10$ calculate the corresponding power of the test in (i).

(iii) Find the size α GLRT for the problem,

$$H_0 : \quad (\theta_1, \theta_2) = (0, 0)$$
$$H_1 : \quad (\theta_1, \theta_2) = (1, 0) \text{ or } (\theta_1, \theta_2) = (0, 1).$$

(iv) For the size $\alpha = 0.10$ calculate the power of GLRT when $(\theta_1, \theta_2) = (0, 1)$.

Q3. Let Y be a number of independent tosses of a coin (with probability of head p, $0 < p < 1$) until the first head occurs (included). Assume the uniform prior for p.

(i) Find the Bayes estimator of p under square loss.

(ii) Does the Bayes estimator (under square loss) of p differ from the moment estimator of p ?

TABLE I. Area Φ (z) under the normal density to the left of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Auxiliary table of quantiles of the normal d.f.

$1-\alpha$	$z(1-\alpha)$	$1-\alpha$	$z(1-\alpha)$	$1-\alpha$	$z(1-\alpha)$
.50	0	.91	1.341	.995	2.576
.55	.126	.92	1.405	.999	3.090
.60	.253	.93	1.476	.9995	3.291
.65	.385	.94	1.555	.9999	3.719
.70	.524	.95	1.645	.99995	3.891
.75	.674	.96	1.751	.99999	4.265
.80	.842	.97	1.881	.999995	4.417
.85	1.036	.98	2.054	.999999	4.753
.90	1.282	.99	2.326	.9999999	5.199