MASTER'S COMPREHENSIVE EXAM IN
REAL ANALYSIS (Math 600)
January 2015

Do any three problems. Show all work. Each problem is worth ten points.

1. Let A and B be subsets of R^n with A bounded. Let $f : R^n \to R^n$ be continuous. Prove the following statements:
   (a) The closure $\overline{A}$ of A is compact.
   (b) $f(\overline{B}) \subseteq \overline{f(B)}$.
   (c) $f(\overline{A}) = \overline{f(A)}$.

2. (i) Define connectedness of a set in a metric space.
   (ii) Show that the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in R^2 is connected.
   (iii) Show that the function $f(x, y) = 3x^3 - 5y^3 - 4$ has a zero on the above ellipse.

3. (a) State the Weierstrass M-test for uniform convergence of a series of real valued functions on a metric space.
   (b) Show that the series $\sum_1^\infty \frac{x^n \sin nx}{n}$ converges for all $x \in (-1, 1)$ and that the sum is continuous on $(-1, 1)$.
   (c) Justify term-by-term integration and differentiation of the given series.

4. (a) State a necessary and sufficient condition for a set to be compact in the metric space $C[0, 1]$ consisting of all real valued continuous functions on $[0, 1]$ endowed with the supremum norm metric.
   (b) If K_1 and K_2 are compact subsets of $C[0, 1]$, show that the set

$$K = \{fg \mid f \in K_1, g \in K_2\}.$$

   is also compact in $C[0, 1]$.

5. State the definition of (Fréchet) derivative of a mapping from one normed linear space to another.

- For the function $f : R^2 \to R$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{|x|}+\sqrt{|y|}}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

  compute the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ at the origin. Is f differentiable at the origin? Justify your answer.