Masters Comprehensive Exam in Matrix Analysis (Math 603)
January 2015

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

1. a) Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $n \times m$ matrix and suppose that A is a $r \times r$ submatrix of M with rank r. Show that $rank(M) = r$ if and only if $D = CA^{-1}B$.

b) Let A and B be two $n \times n$ matrices. Prove that

$$\max\{\dim \text{ Null}(A), \dim \text{ Null}(B)\} \leq \dim \text{ Null}(AB) \leq \dim \text{ Null}(A) + \dim \text{ Null}(B).$$

c) Let A be an $n \times n$ matrix. Suppose that at least $n^2 - n + 1$ entries of A are zeros, prove that $rank(A) < n$. In this case, what is the maximum possible value for $rank(A)$?

2. a) Let A be an $m \times m$ matrix. Given $\epsilon > 0$, prove there exists an invertible $m \times m$ matrix $\tilde{A}$ such that $||A - \tilde{A}||_2 < \epsilon$.

b) If $A,\ B,\ C$ and D are all $m \times m$ matrices and $AB = BA$, prove that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(DA - CB)$$

3. a) Let P be a projector (that is, $P^2 = P$) other than identity and 0. Prove that $||P||_2 = ||(I - P)||_2$.

b) Let P be a projector. Prove that $P = BC^T$ for some $n \times r$ matrices $B,\ C$ such that $C^T B = I_r$.

4. If two real-valued $n \times n$ matrices $A,\ B$ are symmetric positive semidefinite, prove that $trace(AB) \geq 0$ and the eigenvalues of AB are all real.

5. a) Let A be a real-valued $n \times n$ symmetric matrix and let $f(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$ for $\mathbf{x} \in R^n \setminus \{0\}$. Show that $\frac{d}{dt}\,|_{t=0}\, f(\mathbf{x} + t\mathbf{y}) = 0$ for all $\mathbf{y} \in R^n$ implies that $\frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$ is an eigenvalue of A. (Hint: Expand $f(\mathbf{x} + t\mathbf{y})$.)

b) Let$A$ be a real-valued $n \times n$ symmetric matrix and let $C = A + \mathbf{z}\mathbf{z}^T$ for some $\mathbf{z} \in R^n$. Let λ_1 and λ_n be the largest and the smallest eigenvalues of A and ν_1 and ν_n be the largest and the smallest eigenvalues of C. Prove that $\lambda_1 \leq \nu_1$ and $\lambda_n \leq \nu_n$. (Hint: explore the connections between λ_1 and λ_n and $f(\mathbf{x})$).