Masters Comprehensive Exam in Matrix Analysis (Math 603)

August 2015

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

1. Let A and B be two (different) $n \times n$ real matrices such that $R(A) = R(B)$, where $R(\cdot)$ denotes the range of a matrix.

(1) Show that $R(A + B)$ is a subspace of $R(A)$.

(2) Is it always true that $R(A + B) = R(A)$? If so, prove it; otherwise, give a counterexample.

2. Solve the following problems.

(1) Show that an $n \times n$ real matrix A has rank one if and only if there exist two nonzero column vectors $u, v \in \mathbb{R}^n$ such that $A = uv^T$.

(2) Let A and B be two real $n \times n$ rank-one matrices. Show that either $AB = 0$ or AB has rank one.

(3) Let A and B be two real $n \times n$ rank-one matrices with $R(A) \neq R(B)$. Suppose $n \geq 3$. Show that $A + B$ is singular, and determine the largest possible rank of $A + B$.

3. Let $x = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \cdots \\ y_n \end{bmatrix}$ be two column n-dimensional vectors. Denote $\max_i y_i$ by $\overline{y}$, and $\min_i y_i$ by $\underline{y}$. If $\sum_{i=1}^n x_i = 0$, show that $\left| x^T y \right| \leq \frac{1}{2} |x|_1 \left(\overline{y} - \underline{y} \right)$, where $|x|_1 = \sum_{i=1}^n |x_i|$.

4. Prove the following two statements regarding the trace:

(a) Let A be a nonsingular matrix in $\mathbb{R}^{n \times n}$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote its n real eigenvalues. Show that

$$\sum_{i=1}^n |\lambda_i|^2 \leq tr(AA^T).$$

(b) Let $A_1, A_2, \cdots, A_m$ be m symmetric matrices in $\mathbb{R}^{n \times n}$. Suppose that $\sum_{j=1}^m A_j^2 = 0$, then $A_1 = A_2 = \cdots = A_m = 0$.

5. Prove the following two statements:

(a) If A and B are two positive semidefinite matrices in $\mathbb{R}^{n \times n}$, then $tr(AB) \geq 0$. If, in addition, $tr(AB) = 0$, then $AB = BA = 0$.

(b) Let $A_1, A_2, \cdots, A_m$ be m linearly independent symmetric matrices in $\mathbb{R}^{n \times n}$. Let Y and Z be two positive definite matrices in $\mathbb{R}^{n \times n}$. Let M be the matrix in $\mathbb{R}^{m \times m}$ such that

$$M_{ij} = tr(A_i Z A_j Y), \qquad i, j = 1, \cdots, m.$$

Show that M is positive definite. (Hint: show $x^T M x > 0$ for each nonzero column vector x in $\mathbb{R}^m$.)