MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
August 2015

Do any three problems. Show all work. Each problem is worth ten points.

Q1 (a) For a subset of a metric space, provide the definitions of sequential compactness and (open cover) compactness. State how these two concepts are related.

(b) Prove that every compact set in a metric space is closed and bounded.

(c) Is the converse in Part (b) true? Justify your answer.

Q2 (a) Define arcwise (=path) connectedness of a set in a metric space. State a relation between arcwise connectedness and connectedness of a set.

(b) Show that the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ is arcwise connected in $\mathbb{R}^2$.

(c) Is there a non-constant continuous function from $\mathbb{R}^n$ to $\mathbb{Q}$ (= the set of all rational numbers)? Justify.

Q3 Let $g : \mathbb{R} \to \mathbb{R}$ be periodic with period one and on the interval $[0, 1]$ it is given by

$$g(x) = \begin{cases} -2x + 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define f by

$$f(x) := \sum_{0}^{\infty} \frac{g(2^k x)}{2^k}.$$

(a) Show that the series converges uniformly on $\mathbb{R}$ and that f is continuous.

(b) Find the value of $\int_0^1 f(x)\, dx$. [Hint: Drawing the graph of g on $[0, 1]$ may be helpful.]

Q4 Let $C([0, 1])$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ endowed with the supremum norm. Provide the definition of equicontinuity of a subset $K \subset C([0, 1])$.

Let $\phi : [0, 1] \to [0, 1]$ and $\psi : \mathbb{R} \to \mathbb{R}$ be (fixed) continuous maps. Let $K \subset C([0, 1])$ be equicontinuous.

(a) Prove that the set

$$\{f \circ \phi \,|\, f \in K\}$$

is equicontinuous (here $\circ$ denotes function composition).

(b) Suppose that (in addition to being equicontinuous) K is also bounded. Prove that the set

$$\{\psi \circ f \,|\, f \in K\},$$

is also equicontinuous.

Q5 (a) Provide the definition of the (Frechet) derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are normed vector spaces (possibly infinite dimensional).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = |y|^\alpha, \quad 0 \le |y| \le x^2,$$
$$f(x, y) = x^2, \quad \text{otherwise},$$

where $\alpha > 0$. Prove that at $(0, 0)$ the *directional derivatives* of f exist along any $v \in \mathbb{R}^2$ and evaluate these. Find the range of α for which f is (Frechet) differentiable at $(0, 0)$, proving your answer.