PhD COMPREHENSIVE EXAM IN
ORDINARY DIFFERENTIAL EQUATIONS
January 2014

Do any 3 of the following 4 problems. Show all work. Each problem is worth ten points.

Q1. You are given a linear time invariant system $\dot{x} = Ax$ in $\mathbb{R}^4$. A particular trajectory of the system is given by

$$x(t) = e_1 + e_2 t + e_3 e^{-t} + e_4 t e^{-t},$$

for all $t \in \mathbb{R}$, where e_i is the standard basis vector with a 1 in the ith coordinate. Find the matrix A.

Q2. You are given two vector fields f and g on $\mathbb{R}^n$, where f is globally Lipschitz with constant K and g is C^1 on $\mathbb{R}^n$ and satisfies

$$\alpha \equiv \sup_{x \in \mathbb{R}^n} |g(x) - f(x)| < \infty.$$

Let $[0, \beta)$ be the forward maximal interval of existence for g and let $\phi(t, x)$ and $\psi(t, x)$ be the flows of f and g, respectively.

(a) Obtain the estimate

$$|\psi(t, x) - \phi(t, x)| \leq K \int_0^t |\psi(s, x) - \phi(s, x)| ds + \alpha t, \quad t \in [0, \beta).$$

(b) Prove that $\beta = \infty$. *Hint*: Assume $\beta < \infty$ and apply Gronwall's lemma.

Q3. For the three dimensional system

$$\begin{aligned} \dot{x} &= y - x, \\ \dot{y} &= x - y - xz, \\ \dot{z} &= xy - z, \end{aligned}$$

show that

$$L = (x^2 + y^2 + z^2)/2$$

is a Lyapunov function for the equilibrium at origin. Is L a strong or weak Lyapunov function? What does LaSalle's principle say about the asymptotic stability of origin?

Q4. Provide the definition of $\omega(\Gamma)$, the ω-limit set of a trajectory Γ of a dynamical system $\dot{x} = f(x)$ assuming global existence of solutions.

Suppose a trajectory Γ remains in a compact set K for all $t \geq 0$ and that $\omega(\Gamma)$ is a single element set $\{\bar{x}\}$. Prove that the trajectory approaches $\bar{x}$ as $t \to \infty$.