PhD COMPREHENSIVE EXAM IN PARTIAL DIFFERENTIAL EQUATIONS

August 2013

Do any 3 of the 4 problems. Show all work. Each problem is worth ten points.

Q1. Consider the wave equation problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && |x| < \infty,\ t > 0 \\ &u(x,0) = f(x),\ u_t(x,0) = g(x) && |x| < \infty. \end{aligned}$$

Let (a,b) be a fixed open interval and $x_1 > b > a$. Find the time interval during which solution $u(x,t)$ satisfies $u(x_1,t) > 0$, if

(a) $g \equiv 0$ and $f(x) = \begin{cases} > 0 & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$

(b) $f \equiv 0$ and $g(x) = \begin{cases} > 0 & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$

Q2. (a) Let $\Omega \subset \mathbb{R}^n$, $u \in L^1(\Omega)$, α being a multiindex. Define the α-th weak derivative of u. Then define the Sobolev space $W^{k,p}(\Omega)$.

(b) Prove that if u has a weak derivative in Ω, it is unique up to measure zero.

(c) Give an example of a function in $W^{1,2}(\Omega)$ that is **not** in $W^{2,2}(\Omega)$, where $\Omega = (0,1)$.

Q3. Consider the first-order problem $xu_x + yu_y - (x+y)u = 0$, $u(1,y) = 1$. The solution exists on the half plane $x > 0$.

(a) What are the characteristic equations for this PDE?

(b) Find the solution $u(x,y)$.

Q4. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open connected domain with smooth boundary. Let L be defined by $Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + u$, where $a^{ij} \in L^\infty(\Omega)$.

(a) What does it mean for L to be uniformly elliptic in Ω?

(b) Let u be a solution to the problem $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$. Prove $u \equiv 0$ in Ω.