**MASTER'S COMPREHENSIVE EXAM IN**
**Math 600 -REAL ANALYSIS**
**August 2013**

*Do any three problems. Show all work. Each problem is worth ten points.*

1. Let $(M, d)$ be a metric space. Given a set $A \subseteq M$ and $\epsilon > 0$, the $\epsilon$-*dilation* of $A$ is defined by

$$A_\epsilon = \{y \in M \,:\, \exists x \in A \text{ such that } d(x, y) < \epsilon\}.$$

   (i) Show that $A_\epsilon$ is an open set.

   (ii) Show that

$$\bigcap_{\epsilon>0} A_\epsilon$$

   is the closure of $A$.

2. (i) Define 'sequential compactness' of a set in a metric space. Is this equivalent to the (open cover) compactness? Answer 'yes' or 'no'.

   (ii) Show that if $A$ and $B$ are compact in $(M, d)$, then $A \times B$ is compact in $(M \times M, \rho)$, where $\rho$ is the product metric on $M \times M$ defined by $\rho((x, y),(u, v)) = d(x, u) + d(y, v)$ is sequentially compact.

   (iii) If $K$ is compact in $R^n$ (with the usual metric), show that the set $\{tx + (1 - t)y : x, y \in K, 0 \leq t \leq 1\}$ is also compact.

3. (i) State the definition of connectedness of a set in a metric space.

   (ii) What is known about the continuous image of a connected set?

   (iii) Suppose $C$ is a nonempty connected set in a metric space $(M, d)$ such that every element of $M$ can be joined to some element of $C$ by an arc in $M$. Show that $M$ is also connected. [Recall that an arc in $(M, d)$ joining $x$ and $y$ is a continuous function $\phi : [0, 1] \to (M, d)$ such that $\phi(0) = x$ and $\phi(1) = y$.]

4. (i) State the definition of uniform convergence of a sequence of real valued functions on a set $A \subseteq R$.

(ii) Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$$

on $R$. Show that the series converges uniformly and absolutely on every interval of the form $[-a, a]$ where $a > 0$.

(iii) Discuss the continuity and differentiability of the sum on $R$.

5. (i) State the definition of differentiability for a map $f : R^n \to R^m$ at a point.

(ii) Suppose that $f : R^n \to R$ is differentiable with $||Df(c)|| \leq L$ for all $c \in R^n$, where $Df(c)$ denotes the derivative of $f$ at $c$. Show that for all $a, b \in R^n$, $|f(b) - f(a)| \leq L||b - a||$. [Hint: Consider $g : [0, 1] \to R$, $g(t) = f((1 - t)a + tb))$ and apply the calculus mean value theorem.]

(iii) Verify Item $(ii)$ for the function $f : R^n \to R$ defined by $f(x) = \sum_1^n \sin^2 x_i$.